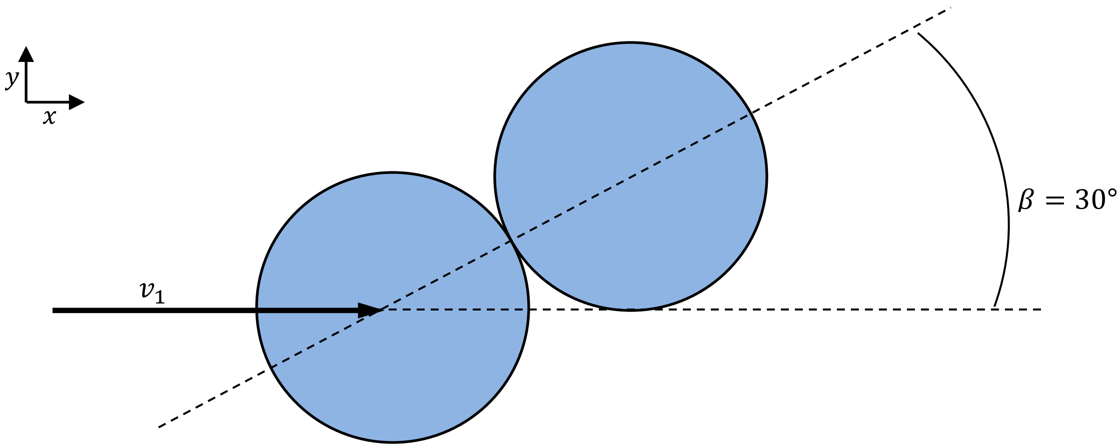
INFOMGP - GAME PHYSICS

EXERCISES LECTURE 7

EXERCISE 7.1

Assuming the following objects, in which direction and with what speed do the two balls move after the collision?
The left ball speed is 10 m/s, the coefficients of restitution are 0.5, and the two masses are 1 kg.



First we identify the line of action which is the normal at the contact points:

$$n = \begin{pmatrix} \cos(30°) \\ \sin(30°) \\ 0 \end{pmatrix}$$

The direction perpendicular to this normal is also required. This is given by:

$$n_\perp = p = \begin{pmatrix} -\sin(30°) \\ \cos(30°) \\ 0 \end{pmatrix}$$

The velocities of object 1 and 2 before collision are:

$$v_{1-} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} \text{ and } v_{2-} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We have the equations of the conservation of momentum and coefficient of restitution:

$$m_1 v_{1-} + m_2 v_{2-} = m_1 v_{1+} + m_2 v_{2+} \text{ and } C_R = -\frac{(v_{1+} - v_{2+}) \cdot n}{(v_{1-} - v_{2-}) \cdot n}$$

As the mass of the two objects are equal and the velocity of the second ball is zero, we can simplify the equations to:

$$\begin{cases} v_{1-}^n = v_{1+}^n + v_{2+}^n \\ v_{1-}^p = v_{1+}^p + v_{2+}^p \\ C_R = -\dfrac{v_{1+}^n - v_{2+}^n}{v_{1-}^n} \end{cases}$$

We note that the second ball will move in the direction of the line of action after the collision, *i.e.* $v_{2+}^p = 0$. Thus, we have:

$$\begin{cases} v_{1-}^n = v_{1+}^n + v_{2+}^n \\ v_{1-}^p = v_{1+}^p \\ -v_{1-}^n C_R = v_{1+}^n - v_{2+}^n \end{cases}$$

This can be simplified by substituting the first equation into the last:

$$\begin{cases} v_{1-}^p = v_{1+}^p \\ -v_{1-}^n C_R = v_{1+}^n - (v_{1-}^n - v_{1+}^n) \end{cases}$$

Solving the last equation for v_{1+}^n gives us:

$$v_{1+}^n = \frac{(1 - C_R)}{2} v_{1-}^n$$

Therefore the final velocity of the first ball is given by:

$$\begin{cases} v_{1+}^n = \dfrac{(1 - C_R)}{2} v_{1-}^n \\ v_{1+}^p = v_{1-}^p \end{cases}$$

Then the velocity of the second ball can be found as:

$$v_{2+}^n = v_{1-}^n - v_{1+}^n = v_{1-}^n - \frac{(1 - C_R)}{2} v_{1-}^n = \left(1 - \frac{(1 - C_R)}{2}\right) v_{1-}^n$$

If we calculate the velocities in the local collision reference frame, we have:

$$v_{1-}^n = v_{1-} \cdot n = 10 \cos(30°) \approx 8.66 \text{ m/s}$$

and

$$v_{1-}^p = v_{1-} \cdot p = -10 \sin(30°) = -5 \text{ m/s}$$

Therefore we have the velocities after collision:

$$v_{1+}^n = \tfrac{(1 - C_R)}{2} v_{1-}^n = \tfrac{1}{4} \times 8.66 = 2.165 \text{ m/s}$$

$$v_{1+}^p = v_{1-}^p = -5 \text{ m/s}$$

$$v_{2+}^n = \left(1 - \tfrac{(1 - C_R)}{2}\right) v_{1-}^n = \tfrac{3}{4} \times 8.66 = 6.495 \text{ m/s}$$

$$v_{2+}^p = 0$$

And finally we calculate them in the world reference frame, so we have:

$$v_{1+} = v_{1+}^n \cdot n + v_{1+}^p \cdot p = \begin{pmatrix} 2.165\cos(30°) - (-5)\sin(30°) \\ 2.165\sin(30°) + (-5)\cos(30°) \\ 0 \end{pmatrix} \approx \begin{pmatrix} 4.37 \\ -3.25 \\ 0 \end{pmatrix}$$

$$v_{2+} = v_{2+}^n \cdot n + v_{2+}^p \cdot p = \begin{pmatrix} 6.495\cos(30°) \\ 6.495\sin(30°) \\ 0 \end{pmatrix} \approx \begin{pmatrix} 5.625 \\ 3.25 \\ 0 \end{pmatrix}$$

A second more "direct" way to solve this problem (the one used in a physics engine) makes use of the scaling factor of the impulse magnitude j as seen in the lecture.

First we still need to identify the line of action which is the normal at the contact points:

$$n = \begin{pmatrix} \cos(30°) \\ \sin(30°) \\ 0 \end{pmatrix}$$

We know that the outgoing velocities can be calculated as:
$v_{1+} = v_{1-} + \frac{j_1}{m_1} n$ and $v_{2+} = v_{2-} - \frac{j_2}{m_2} n$

And as the two C_R are equal, we have the scaling factors:

$$j_1 = j_2 = -\frac{(1 + C_R)(v_{1-} - v_{2-})}{\frac{1}{m_1} + \frac{1}{m_2}} \cdot n = -\frac{(1 + 0.5)}{2} \left[\begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} \cos(30°) \\ \sin(30°) \\ 0 \end{pmatrix} = -7.5\cos(30°)$$

So finally we can calculate the outgoing velocities as follows:

$$v_{1+} = v_{1-} + \frac{j_1}{m_1} n = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -7.5\cos(30°)\cos(30°) \\ -7.5\cos(30°)\sin(30°) \\ 0 \end{pmatrix} \approx \begin{pmatrix} 4.37 \\ -3.25 \\ 0 \end{pmatrix}$$

and

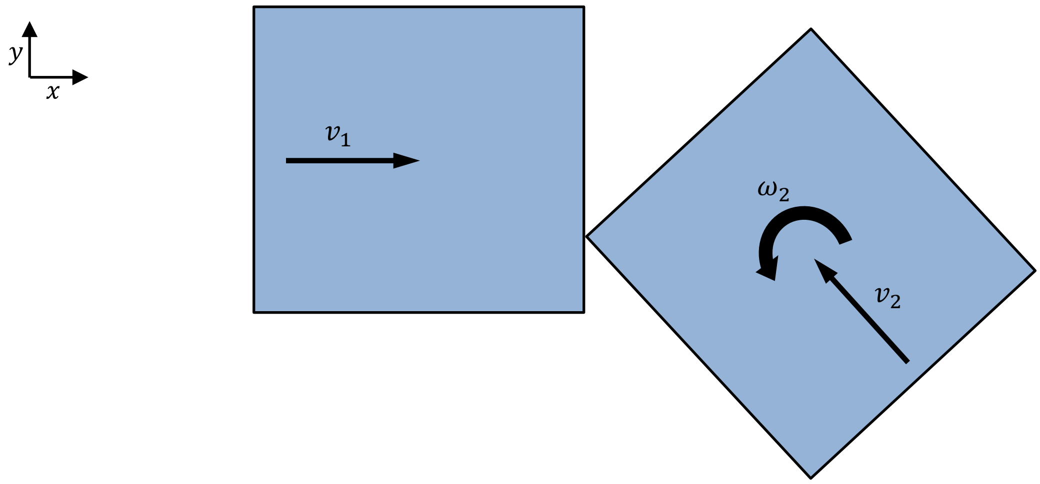
$$v_{2+} = v_{2-} - \frac{j_2}{m_2} n = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} -7.5\cos(30°)\cos(30°) \\ -7.5\cos(30°)\sin(30°) \\ 0 \end{pmatrix} \approx \begin{pmatrix} 5.625 \\ 3.25 \\ 0 \end{pmatrix}$$

Hopefully we find the same result as in the first solution!

EXERCISE 7.2

Assume the following objects moving toward each other with the second box rotating around its center of mass. At what linear and angular velocities will the boxes move after the collision?
The speed of the boxes is 10 m/s, the angular speed of the second box is 5 rad/s, the coefficients of

restitution are 0.5, the size of the boxes is 2x2x2, the two masses are 1 kg, the velocity of the second box is at 45°and the second box hits the first one at a quarter of its height.



We can first summarize the data we know

$$v_{1-} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} \text{ and } \omega_{1-} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_{2-} = \begin{pmatrix} -5\sqrt{2} \\ 5\sqrt{2} \\ 0 \end{pmatrix} \text{ and } \omega_{2-} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

$$r_1 = \begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} \text{ and } r_2 = \begin{pmatrix} -\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

$$n = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$I_1 = I_2 = \begin{bmatrix} \frac{1}{12} m_1(w^2 + h^2) & 0 & 0 \\ 0 & \frac{1}{12} m_1(d^2 + h^2) & 0 \\ 0 & 0 & \frac{1}{12} m_1(d^2 + w^2) \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

From these data, we can start by calculating the velocities of the respective collision points p_1 and p_2:

$$v_{p_{1-}} = v_{1-} + \omega_{1-} \times r_{1-} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

$$v_{p_{2-}} = v_{2-} + \omega_{2-} \times r_{2-} = \begin{pmatrix} -5\sqrt{2} \\ 5\sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} \times \begin{pmatrix} -\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

Then we determine the velocities in the line of action on the points of impact, by taking the inner product of the collision normal with the respective velocities:

$$v_{1-}^n = n \cdot v_{p_{1-}} = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}10\\0\\0\end{pmatrix} = 10$$

$$v_{2-}^n = n \cdot v_{p_{2-}} = \begin{pmatrix}1\\0\\0\end{pmatrix} \cdot \begin{pmatrix}-5\sqrt{2}\\0\\0\end{pmatrix} = -5\sqrt{2}$$

Next we can calculate the impulse that needs to be applied to the left box, using the equations of impulse/momentum and coefficient of restitution:

$$j = \frac{-(1 + C_R)(v_{1-} - v_{2-}) \cdot n}{\left(\frac{1}{m_1} + \frac{1}{m_2}\right) + \left[\left(I_1^{-1}(r_1 \times n)\right) \times r_1 + \left(I_2^{-1}(r_2 \times n)\right) \times r_2\right] \cdot n}$$

Let's look at the nominator of the fraction, we have:

$$-(1 + C_R)(v_{1-} - v_{2-}) \cdot n = -(1 + 0.5)\left(10 + 5\sqrt{2}\right) \approx -25.6 \text{ m/s}$$

Then let's look at the denominator of the fraction, and first the part related to object 1:

$$\left[\left(I_1^{-1}(r_1 \times n)\right) \times r_1\right] \cdot n = \left[\left[\left(\begin{pmatrix}1\\-0.5\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix}\right) / \begin{pmatrix}\frac{2}{3} & 0 & 0\\ 0 & \frac{2}{3} & 0\\ 0 & 0 & \frac{2}{3}\end{pmatrix}\right] \times \begin{pmatrix}1\\-0.5\\0\end{pmatrix}\right] \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= \left[\left[\begin{pmatrix}0\\0\\0.5\end{pmatrix} / \begin{pmatrix}\frac{2}{3} & 0 & 0\\ 0 & \frac{2}{3} & 0\\ 0 & 0 & \frac{2}{3}\end{pmatrix}\right] \times \begin{pmatrix}1\\-0.5\\0\end{pmatrix}\right] \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3/8\\3/4\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} = 3/8$$

We can do the same for the part related to the object 2:

$$\left[\left(I_2^{-1}(r_2 \times n)\right) \times r_2\right] \cdot n = \left[\left[\left(\begin{pmatrix}-\sqrt{2}\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\0\end{pmatrix}\right) / \begin{pmatrix}\frac{2}{3} & 0 & 0\\ 0 & \frac{2}{3} & 0\\ 0 & 0 & \frac{2}{3}\end{pmatrix}\right] \times \begin{pmatrix}-\sqrt{2}\\0\\0\end{pmatrix}\right] \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$= \left[\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} / \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix} \right] \times \begin{pmatrix} -\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

We can now combine them back to calculate j:

$$j = \frac{-25.6}{1 + 1 + \frac{3}{8} + 0} = -10.8$$

Thanks to this value we can finally calculate the velocities after collision:

$$v_{1+} = v_{1-} + \frac{j}{m_1} n = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} - 10.8 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.8 \\ 0 \\ 0 \end{pmatrix}$$

$$v_{2+} = v_{2-} - \frac{j}{m_2} n = \begin{pmatrix} -5\sqrt{2} \\ 5\sqrt{2} \\ 0 \end{pmatrix} + 10.8 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \approx \begin{pmatrix} 3.7 \\ 7.1 \\ 0 \end{pmatrix}$$

$$\omega_{1+} = \omega_{1-} + I_1^{-1}(r_1 \times (j \cdot n)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \left[\begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} \times \begin{pmatrix} -10.8 \\ 0 \\ 0 \end{pmatrix} \right] / \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -8.1 \end{pmatrix}$$

$$\omega_{2+} = \omega_{2-} - I_2^{-1}(r_2 \times (j \cdot n)) = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} - \left[\begin{pmatrix} -\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -10.8 \\ 0 \\ 0 \end{pmatrix} \right] / \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$